SYDNEY GRAMMAR SCHOOL

2013 Trial Examination

FORM VI
MATHEMATICS 2 UNIT

Thursday 1st August 2013

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 98 boys

Examiner
BDD

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The value of $\displaystyle\int_0^2 \left(6x^2+1\right)\,dx$ is:

(A) 17

(B) 18

(C) 24

(D) 66

QUESTION TWO

The line intersecting the x-axis at $x = -1$ and passing through the point $A(1,-4)$ is represented by which of the following equations?

(A) $x+2y-1=0$

(B) $x+2y+1=0$

(C) $2x-y-2=0$

(D) $2x+y+2=0$

QUESTION THREE

The quadratic equation $2x^2+12x-9=0$ has roots α and β. The value of $\alpha^2\beta+\alpha\beta^2$ is:

(A) -108

(B) -27

(C) 27

(D) 108

Exam continues next page ...

QUESTION FOUR

What is the sum of the first ten terms of the series $96 - 48 + 24 - 12 + \cdots$?

(A) 63·9375

(B) 191·8125

(C) −32·736

(D) 98·208

QUESTION FIVE

Which of the following does $\dfrac{d}{dx}(e^3)$ equal?

(A) $3e^2$

(B) e^3

(C) 0

(D) $\dfrac{1}{4}e^4$

QUESTION SIX

Which of the following statements is INCORRECT?

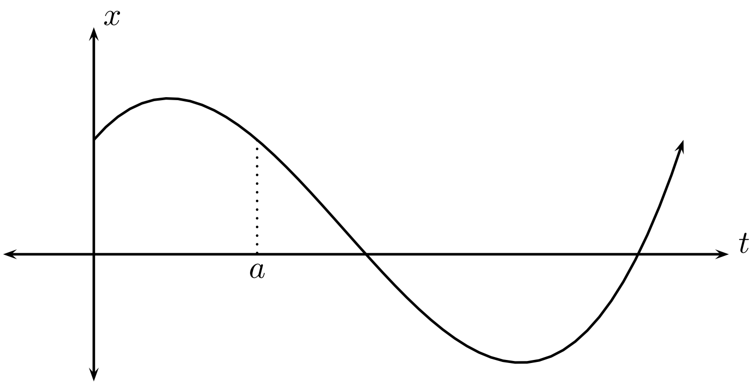
(A) $\log a^n = n \log a$

(B) $\log ab = \log a + \log b$

(C) $\log(a - b) = \dfrac{\log a}{\log b}$

(D) $\log e = 1$

QUESTION SEVEN



A particle's motion is described by the cubic graph above. Which of the following statements is NOT true of the particle at time $t = a$?

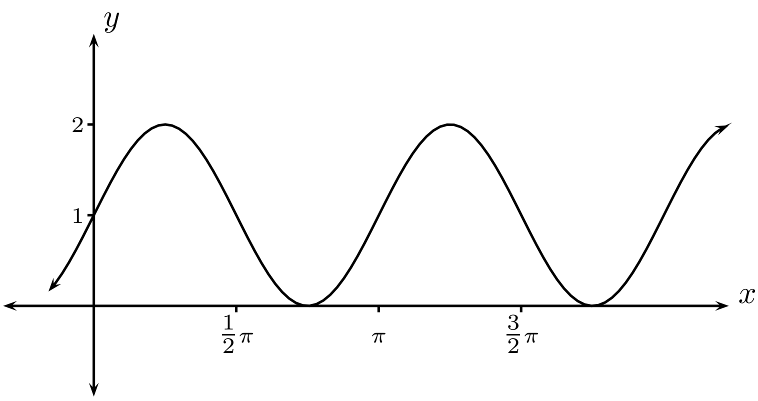
(A) The particle's velocity is negative.

(B) The particle has positive acceleration.

(C) The particle is moving towards the origin.

(D) The particle has returned to its initial position.

QUESTION EIGHT



The equation of the graph sketched above could be:

(A) $y = 1 + \sin 2x$

(B) $y = 1 - \sin 2x$

(C) $y = 1 + 2\sin 2x$

(D) $y = 1 - 2\sin x$

Exam continues next page ...

QUESTION NINE

Which of the following statements is NOT true of the function $y = x^4 + 4x^2$?

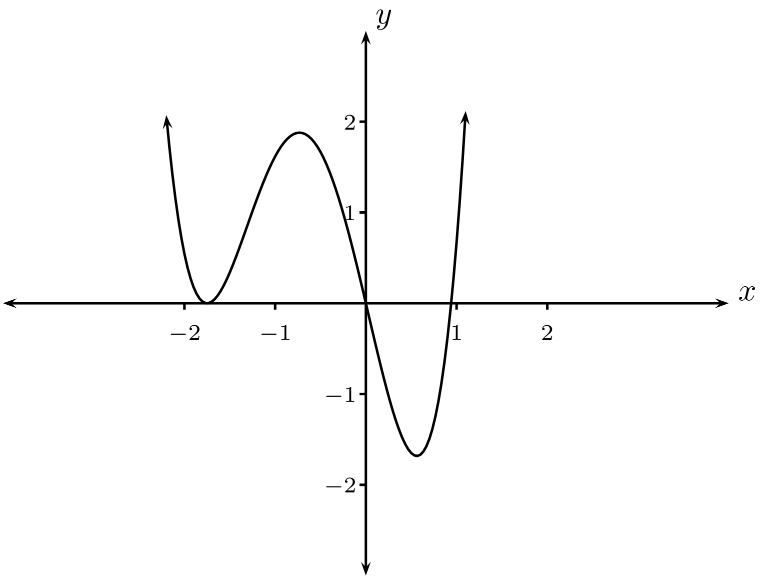
(A) It is even.

(B) It has a single stationary point at $x = 0$.

(C) It has a single x-intercept at $x = 0$.

(D) It has a single point of inflexion at $x = 0$.

QUESTION TEN



The diagram above shows the graph of a function $y = f(x)$. A pupil draws the graph of $y = 2 - |x|$ on the diagram in order to determine the number of solutions to the equation $f(x) = 2 - |x|$. His answer should be:

(A) 0

(B) 1

(C) 2

(D) 4

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find the value of $\dfrac{e^x}{1+x^2}$ when $x = -3$. Give your answer correct to 3 decimal places. **1**

(b) Differentiate:

(i) $y = \cos 2x$ **1**

(ii) $y = \ln(3x+1)$ **1**

(iii) $y = e^{3x}$ **1**

(c) Find the exact value of $\tan \frac{2\pi}{3}$. **1**

(d) Rationalise the denominator of $\dfrac{1}{3-\sqrt{5}}$. **1**

(e) Find the following integrals:

(i) $\displaystyle\int \left(3x^2 + 4x\right)\, dx$ **1**

(ii) $\displaystyle\int \frac{5}{x}\, dx$ **1**

(iii) $\displaystyle\int (2x+1)^5\, dx$ **1**

(f) Find the area of a sector subtending an angle of 6 radians at the centre of a circle of radius 3 cm. **1**

(g) Find the one-hundredth term of the arithmetic sequence with first term 8 and common difference 3. **1**

(h) Solve $2\cos\theta - 1 = 0$, for $0 \leq \theta \leq 2\pi$. **2**

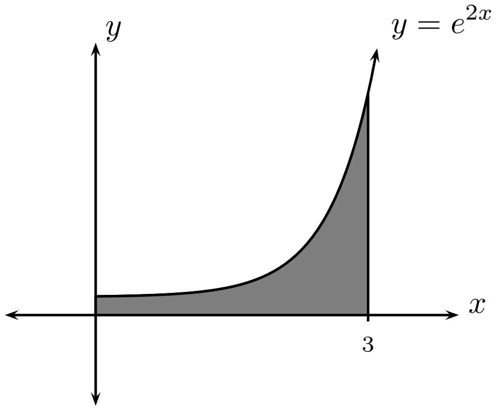
(i) Draw a one-third page sketch of the parabola $x^2 = -8y$, carefully marking the focus and directrix. **2**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

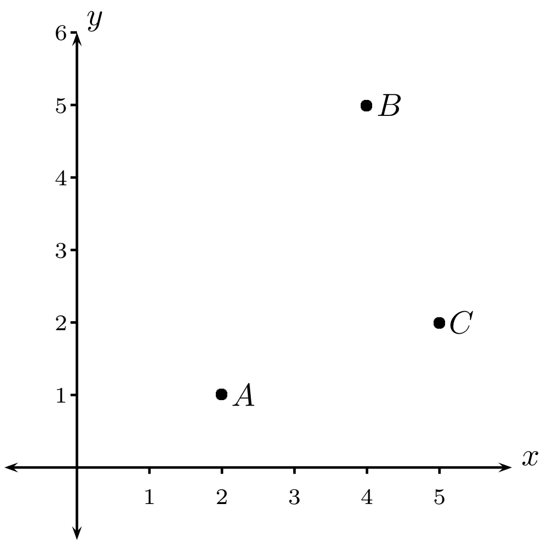
(a) Find the equation of the tangent to $y = x^2 + 4x$ at $x = 1$. **2**

(b) **2**



The graph above shows the area bounded by the curve $y = e^{2x}$, the line $x = 3$ and the coordinate axes. Find the exact shaded area.

(c)



The points $A(2, 1)$, $B(4, 5)$ and $C(5, 2)$ have been marked in the coordinate plane above.

(i) Show that the equation of the line passing through A and B is $2x - y - 3 = 0$. **2**

(ii) Determine the length of interval AB. **1**

(iii) Find the perpendicular distance from the point C to the line AB. **1**

(iv) Hence find the area of triangle ABC. **1**

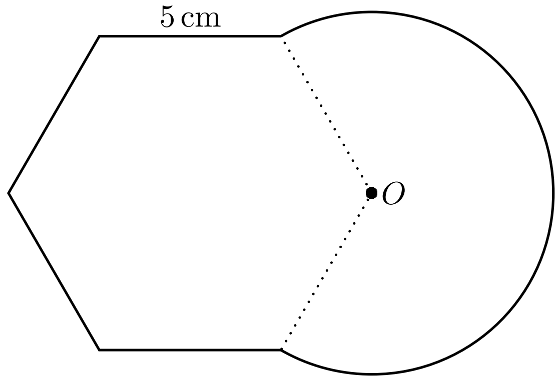
(d) Find the domain and range of $y = \sqrt{2x - 6}$. **2**

(e) Solve the quadratic inequation $x^2 + 2x - 3 < 0$. **2**

Exam continues overleaf ...

QUESTION TWELVE (Continued)

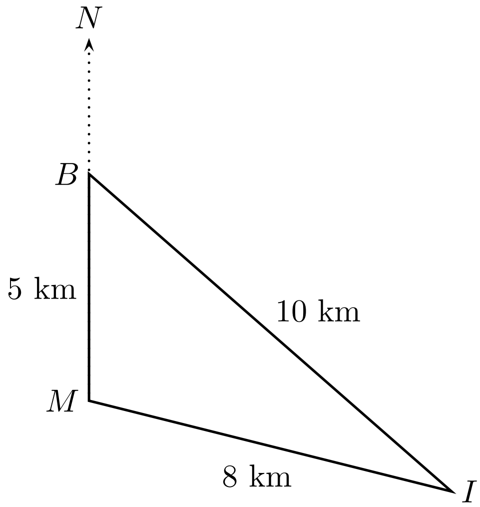
(f) **2**



The diagram above shows a regular hexagon joined to the radii of a sector. The side length of the hexagon is 5 cm. Find the exact perimeter of the resulting shape.

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



An island I, a buoy B and the mainland M lie on the vertices of a triangle, as in the diagram above. The distance from M to B is 5 km, from B to I is 10 km and from I to M is 8 km. The buoy is directly north of the mainland.

(i) Use the cosine rule to find $\angle MBI$, correct to the nearest minute. **2**

(ii) What is the true bearing of the island from the buoy? **1**

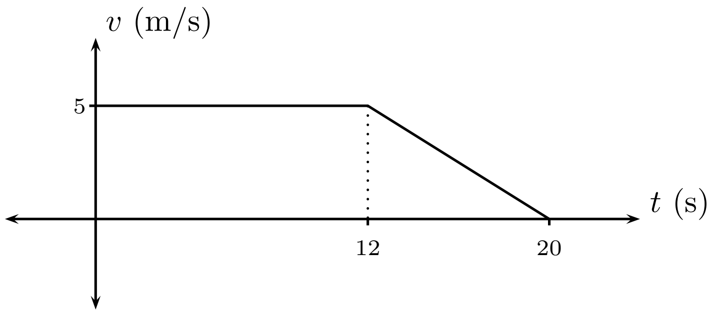
(b) Differentiate $y = \dfrac{e^{5x} + 1}{e^x}$. **2**

Exam continues next page ...

QUESTION THIRTEEN (Continued)

(c) Consider the region bounded by the curve $y = \sec x$, the x-axis and the lines $x = 0$ and $x = \dfrac{\pi}{3}$. Find the volume obtained by rotating this region about the x-axis. **2**

(d) **1**



The velocity-time graph of a particle is shown above. Find the distance travelled in the first 20 seconds.

(e) A particle travelling in one dimension has velocity function $v = 6 - 2t$, where v is in metres per second and t is in seconds. The particle is initially seven metres to the right of the origin. Assume that the positive direction is to the right.

(i) Find the particle's acceleration function. **1**

(ii) Find the particle's displacement function. **1**

(iii) When is the particle at rest and what is its displacement at this time? **2**

(iv) Draw a one-third page sketch of the particle's displacement function, showing the intercepts with the axes and the vertex of this parabola. **2**

(v) What is the total distance travelled over the first eight seconds? **1**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

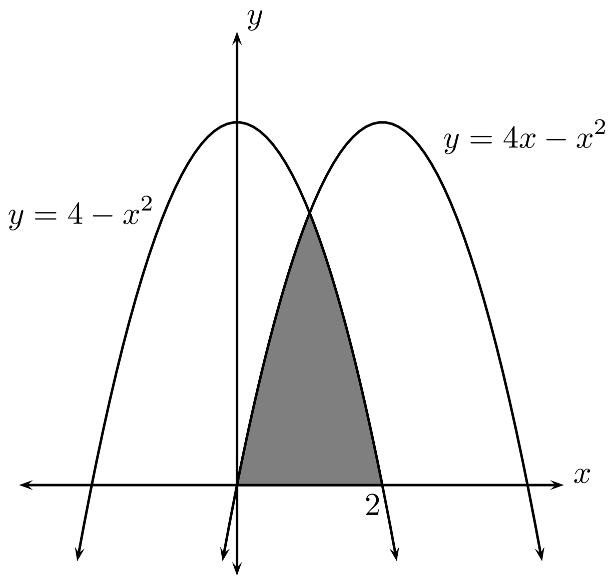
(a) Consider the function $f(x) = x^4 - 4x^3 + 5$.

(i) Find the coordinates of the stationary points of $y = f(x)$. **3**

(ii) Determine the nature of the stationary points. **2**

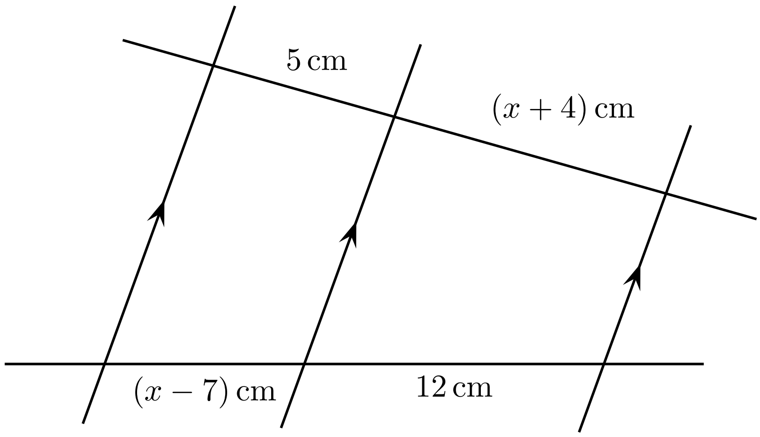
(iii) Sketch the graph of $y = f(x)$, showing the stationary points and y-intercept. You need not find any x-intercepts. **2**

(b) **3**



Find the area of the region shaded in the diagram above.

(c) **2**



Find the value of x in the diagram above, giving a reason.

(d) A point $P(x, y)$ moves so that its distance from $A(6, 1)$ is twice its distance from $B(-3, 4)$.

(i) Show that the locus of P is a circle. **2**

(ii) Find the centre and radius of the circle. **1**

Exam continues next page ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\cos\theta$ given that $\tan\theta = 7$ and $\sin\theta < 0$. **2**

(b) (i) Show that $3x^2 + 4x + 5$ is positive definite. **1**

(ii) Explain why the function $y = x^3 + 2x^2 + 5x + 7$ is always increasing. **1**

(c) Prove that $\sin\theta\tan\theta + \cos\theta = \sec\theta$. **2**

(d) Prove that $f(x) = \dfrac{2x}{x^2+1}$ is an odd function. **1**

(e) (i) Differentiate xe^x. **1**

(ii) Hence find $\displaystyle\int xe^x\,dx$. **1**

(f) The rate of elimination $\dfrac{dQ}{dt}$ of a drug by the kidneys is given by the equation

$$\frac{dQ}{dt} = -kQ$$

where k is a constant and Q is the quantity of drug present in the blood. In this question, t is measured in minutes and Q in milligrams.

(i) Show that $Q = Q_0e^{-kt}$ satisfies the equation $\dfrac{dQ}{dt} = -kQ$. **1**

(ii) The initial quantity of drug present was measured to be 100 mg and at time $t = 20$ minutes, the quantity was 74 mg. Find the values of Q_0 and k. Give k correct to five decimal places and Q_0 to the nearest mg. **2**

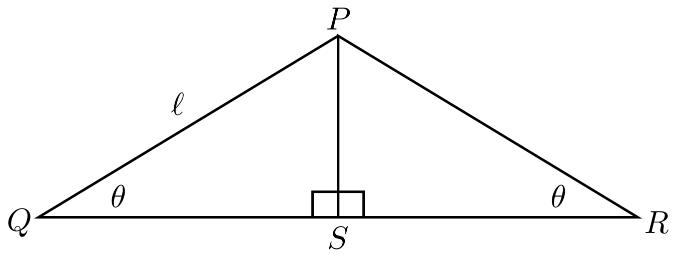
(iii) What is the initial rate of elimination of the drug? Give your answer correct to one decimal place. **1**

(iv) How long is it until only half the original quantity of drug remains? Give your answer correct to the nearest minute. **2**

The exam continues on the next page

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) In the diagram below, $\angle PQS = \angle PRS = \theta$ and $PQ = \ell$.



(i) Prove that $\triangle PQS \equiv \triangle PRS$. **1**

(ii) Give a reason why $QS = RS$. **1**

(iii) Show that $QR = 2\ell \cos\theta$. **1**

(iv) Show that the area of $\triangle PQR$ is given by **1**

$$A = \ell^2 \cos\theta \sin\theta.$$

(v) Use calculus to find the value of θ that gives the maximum area of $\triangle PQR$. **3**

(b) A university student is planning to use a cash account containing \$50 000 to help fund his expenses. The account earns interest at 6% per annum, compounded monthly. At the end of each month interest is added to the account balance and then the student withdraws \$1500. Let A_n be the amount of money remaining in the account at the end of the nth month, following the student's withdrawal.

(i) Find an expression for A_1. **1**

(ii) Find expressions for A_2 and A_3. **2**

(iii) After how many months will the account have a balance of zero dollars? Give your answer to the nearest month. **2**

(c) Solve the following equation, for $0 \le \theta \le 2\pi$: **3**

$$3\sin^2\theta + 3\cos^2\theta + 3\tan^2\theta + 3\cot^2\theta + 3\sec^2\theta + 3\operatorname{cosec}^2\theta = 29$$

End of Section II

END OF EXAMINATION

B L A N K   P A G E

The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \, , \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} \, , \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE : } \; \ln x = \log_e x, \; x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

2 Unit Trial HSC Solutions 2013

MCQ

1. B
2. D
3. C
4. A
5. C
6. C
7. B
8. A
9. D
10. D

QUESTION 11

(a) $\dfrac{e^{-3}}{1+(-3)^2} \doteqdot 0.005$.

(b) (i) $\dfrac{dy}{dx} = -2\sin 2x$

(ii) $\dfrac{d}{dx}\ln(3x+1) = \dfrac{3}{3x+1}$

(iii) $\dfrac{d}{dx}e^{3x} = 3e^{3x}$

(c) $\tan\dfrac{2\pi}{3} = \tan\left(\pi - \dfrac{\pi}{3}\right)$

$= -\tan\dfrac{\pi}{3}$

$= -\sqrt{3}$.

(d) $\dfrac{1}{3-\sqrt{5}} = \dfrac{1}{3-\sqrt{5}} \times \dfrac{3+\sqrt{5}}{3+\sqrt{5}}$

$= \dfrac{3+\sqrt{5}}{9-5}$

$= \dfrac{3+\sqrt{5}}{4}$

(e) (i) $\int (3x^2+4x)\,dx = x^3+2x^2+c$

(ii) $\int \dfrac{5}{x}\,dx = 5\log x + c$

(iii) $\int (2x+1)^5\,dx = \dfrac{(2x+1)^6}{12} + c$

(f) $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \cdot 3^2 cm^2 \cdot 6$

$= 27 \ cm^2$ ✓

(g) $t_n = 8 + 3(n-1)$

so $t_{100} = 8 + 3 \times 99$

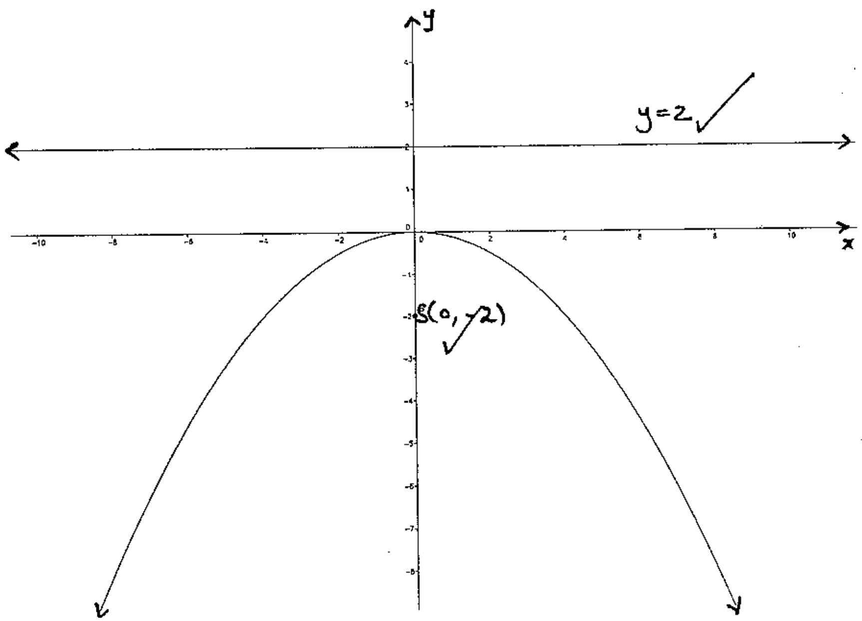
$= 305$. ✓

(h) $2\cos\theta - 1 = 0 , \quad 0 \le \theta \le 2\pi$.

$\cos\theta = \frac{1}{2}$

so $\theta = \frac{\pi}{3}$ ✓ or $\frac{5\pi}{3}$ ✓

(i)



QUESTION 12

(a) For $y = x^2 + 4x$, $\frac{dy}{dx} = 2x + 4$.

At $x = 1$, $\frac{dy}{dx} = 6$ & $y = 5$.

Point : $(1,5)$; gradient : 6

Equation of tangent : $y - 5 = 6(x-1)$

$\therefore y = 6x - 1$

(b) Area given by $\int_0^3 e^{2x}\,dx = \frac{1}{2} e^{2x} \Big|_0^3$

$= \frac{1}{2}\left(e^6 - 1\right)$

(c) $A(2,1)$ $B(4,5)$ $C(5,2)$

(i) Eq^n of line: $y - 1 = \frac{5-1}{4-2}(x-2)$

$y = 2x - 3$

so $2x - y - 3 = 0$, as required.

OR substitute coordinates of A, B to show points lie on the line.

(ii) $AB^2 = (4-2)^2 + (5-1)^2$

$= 4 + 16$

$= 20$

so $AB = 2\sqrt{5}$

(iii) $d_\perp = |2(5) - (2) - 3| \div \sqrt{4+1}$

$= \frac{|5|}{\sqrt{5}}$

$= \sqrt{5}$

(iv) $A_{\Delta ABC} = \frac{1}{2}(AB)\, d_\perp$

$= \frac{1}{2} \cdot 2\sqrt{5} \cdot \sqrt{5}$

$= 5\ u^2$

(d) For $\sqrt{2x-6}$, domain is all real x such that

$2x - 6 \geq 0$; i.e. all real x : $x \geq 3$.

The function $\sqrt{2x-6}$ is increasing for all $x \geq 3$.

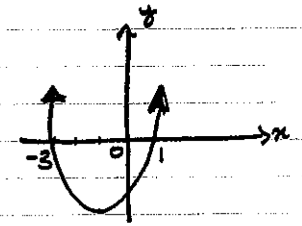
Hence its minimum value lies at the lower bound of its domain $(x = 3)$:

$\min \sqrt{2x-6} = \sqrt{2(3)-6} = 0$.

Since $y = \sqrt{2x-6}$ increases without bound for all $x \geq 3$, the range is then all real $y \geq 0$.

(e) $x^2 + 2x - 3 < 0$

i.e. $(x+3)(x-1) < 0$



Solution is $-3 < x < 1$.

(f) Regular hexagon composed of six congruent equilateral triangles of side length 2cm.

$\therefore$ internal angle is $2 \times 60° = 120°$.

Reflex $\angle$ $\therefore$ $360° - 120° = 240°$

(a) $(\text{Arc length}) = \frac{240°}{360°} \times 2\pi \times 2\text{cm} = \frac{8\pi}{3}$ cm.

(β) (Hexagonal contribution) $= 4 \times 2\text{cm} = 8\text{cm}$.

$\therefore$ (perimeter) $= 8\text{cm} + \frac{8\pi}{3}\text{cm}$

$= 8\left(1+\frac{\pi}{3}\right)\text{cm}$.

QUESTION 13

(a) (i) $8^2 = 5^2 + 10^2 - 2\cdot5\cdot10\cos(\angle MBI)$

$\Rightarrow \cos(\angle MBI) = 0\cdot61$

so $\angle MBI \doteqdot 52°25'$

(ii) True bearing: $180° - 52°25' \doteqdot 127°35'T$

(b) $y = \frac{e^{5x}+1}{e^x}$

$= e^{4x} + e^{-x}$

so $\frac{dy}{dx} = 4e^{4x} - e^{-x}$

(c) $V = \pi\int_0^{\pi/3} y^2\,dx$

$= \pi\int_0^{\pi/3} \sec^2 x\,dx$

$= \pi\Big[\tan x\Big]_0^{\pi/3}$

$= \pi\left[\sqrt{3} - 0\right]$

$= \sqrt{3}\pi\ u^3$.

(d) $|x| = \int_0^{12} v_1(t)\,dt + \int_{12}^{20} v_2(t)\,dt$

$= (12-0)(5) + \frac{1}{2}(20-12)(5)$

$= 60 + 20$

$= 80 \text{ m}$.

(e) $v = 6-2t$ m/s; $x(0) = 7$ m.

(i) $a = \frac{dv}{dt} = -2 \text{ m/s}^2$.

(ii) $x = \int 6-2t \; dt$

$= 6t - t^2 + c$

When $t=0$, $x = 7$, so $c = 7$.

$\therefore \; x(t) = 6t - t^2 + 7$.

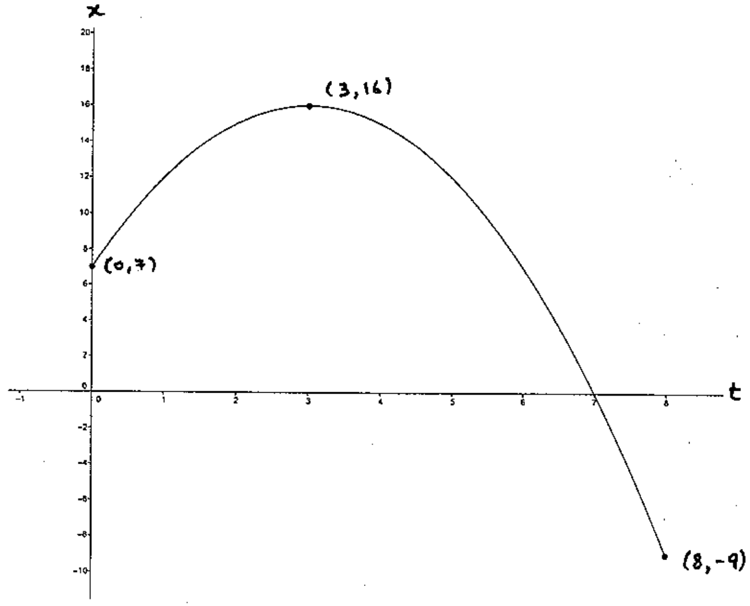
(iii) Particle at rest when $v(t) = 0$. Then

$6-2t = 0 \Rightarrow t = 3$ s.

When $t=3$, $x(3) = 6(3) - (3)^2 + 7$

$= 16$ m.

(iv)



(v) Total distance: from $t=0$ to $t=3$, distance is $16-7 = 9$m;
from $t=3$ to $t=7$, distance is 16 m;
from $t=7$ to $t=8$, distance is 9 m.

Total $\therefore$ 34m.

QUESTION 14

(a) $f(x) = x^4 - 4x^3 + 5$.

(i) Stationary points for x: $f'(x) = 0$.

Now, $f'(x) = 4x^3 - 12x^2$, so solving

$$4x^2(x-3) = 0 \Rightarrow x = 0 \text{ or } x = 3. \checkmark$$

For $x = 0$, $f(0) = 5 \rightarrow (0,5)$ is a stationary point. ✓

For $x = 3$, $f(3) = -22 \rightarrow (3,-22)$ is a stationary point. ✓

(ii) Nature:

$$f''(x) = 12x^2 - 24x$$

$$= 12x(x-2)$$

Now, $f''(0) = 0 \rightarrow$ indeterminate : use table.

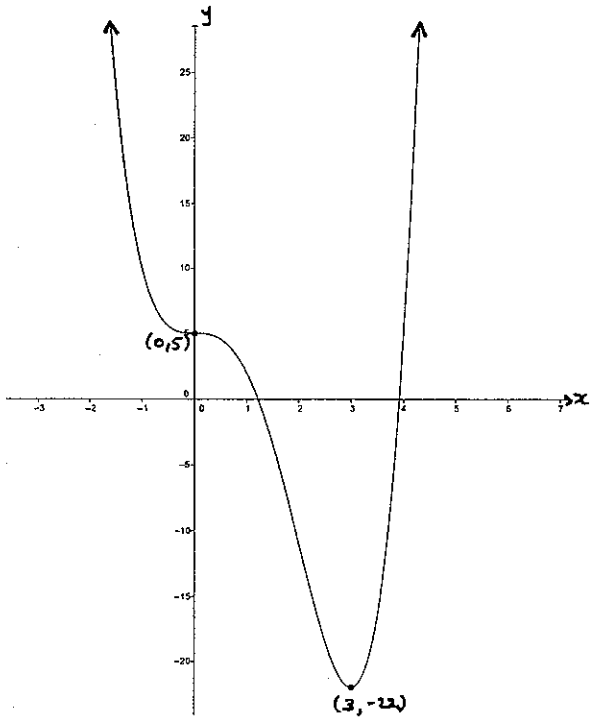
$f''(3) = 36 > 0 \rightarrow$ local minimum at $(3,-22)$. ✓

For $(0,5)$:

x	-1	0	1
$f''(x)$	36	0	-12
concav.	$\cup$	.	$\cap$

$\therefore$ horizontal point of inflexion at $(0,5)$. ✓

(iii)



✓ for correct information from above

✓ right shape

(b) Intersection of graphs occurs for x such that

$$4 - x^2 = 4x - x^2$$

i.e. $x = 4$.

Area, A, is compound, so

$$A = \int_0^1 4x - x^2 \, dx + \int_1^2 4 - x^2 \, dx \quad \checkmark$$

at least one correct integration + substitution without simplification.

$$= 2x^2 - \frac{x^3}{3}\Big|_0^1 + 4x - \frac{x^3}{3}\Big|_1^2$$

$$= \left(2 - \frac{1}{3}\right) - (0) + \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)$$

$$= \frac{10}{3} \text{ or } 3\frac{1}{3} \ u^2 .$$

(c) By ratio of intercepts of transversals on parallel lines,

$$\frac{x+4}{5} = \frac{12}{x-7}$$

with reason (saying only "ratio of intercepts" okay).

so

$$(x-7)(x+4) = 60$$

$$x^2 - 3x - 88 = 0$$

$$(x-11)(x+8) = 0$$

$\therefore$ $x = 11$ cm or $x = -8$cm.

$x = -8$cm does not work. So $x = 11$cm.

(d) (i) (Distance of P to A) is 2 × (distance of P to B)

i.e.

$$\overline{AP} = 2\overline{BP}$$

$$\overline{AP}^2 = 4\overline{BP}^2$$

$$(x-6)^2 + (y-1)^2 = 4\left[(x+3)^2 + (y-4)^2\right]$$

$$x^2 - 12x + 36 + y^2 - 2y + 1 = 4\left[x^2 + 6x + 9 + y^2 - 8y + 16\right]$$

$$= 4x^2 + 24x + 36 + 4y^2 - 32y + 64$$

which simplifies to

$$x^2 + 12x + y^2 - 10y + 21 = 0$$

$$x^2 + 12x + 6^2 + y^2 - 10y + (-5)^2 = -21 + 6^2 + (-5)^2$$

$$(x+6)^2 + (y-5)^2 = 40$$

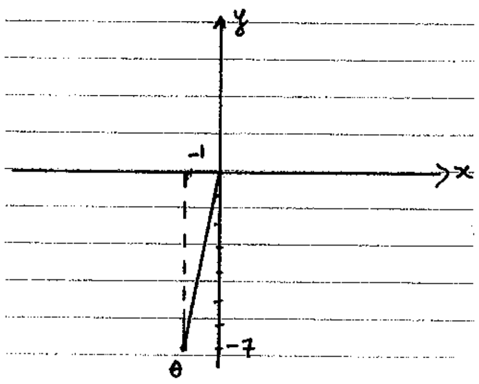
This is the equation of a circle.

(ii) From (i), centre: $(-6, 5)$

radius: $2\sqrt{10}$.

QUESTION 15

(a) Since $\tan\theta > 0$, $\sin\theta < 0$, θ lies in quadrant III.



Now, $\tan\theta = \frac{y}{x}$ for any point (x, y) on the ray given by θ.

Choose, then, $x = -1$, $y = -7$.

Then $r = \sqrt{x^2 + y^2}$

$= \sqrt{1 + 49}$

$= \sqrt{50}$

$= 5\sqrt{2}$ ✓

So $\cos\theta = \frac{x}{r}$

$= \frac{-1}{5\sqrt{2}}$ ✓✓

OR Given $\tan\theta = 7$,

$\tan^2\theta = 49$

$\frac{\sin^2\theta}{\cos^2\theta} = 49$

$\sin^2\theta = 49\cos^2\theta$

$1 - \cos^2\theta = 49\cos^2\theta$

So $\cos^2\theta = \frac{1}{50}$

i.e. $\cos\theta = \pm\frac{1}{5\sqrt{2}}$ ✓

Since θ is in quadrant III,

$\cos\theta = -\frac{1}{5\sqrt{2}}$. ✓

(b) (i) $3x^2 + 4x + 5$ has discriminant

$\Delta = 4^2 - 4(3)(5) < 0$

and coefficient of x^2 is positive. ✓

Hence the graph of $3x^2 + 4x + 5$ lies entirely above the x-axis: the quadratic is positive definite.

(ii) $y = x^3 + 2x^2 + 5x + 7$

so $\frac{dy}{dx} = 3x^2 + 4x + 5$

which, by (i), is positive for all real x;

i.e. $\frac{dy}{dx} > 0$ for all x.

Hence $y = x^3 + 2x^2 + 5x + 7$ is increasing always. ✓

(c) $\sin\theta\tan\theta + \cos\theta = \sin\theta \cdot \frac{\sin\theta}{\cos\theta} + \cos\theta$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta \quad //$$

(d) $f(x)$ is odd if $f(-x) = -f(x)$.

Now $f(x) = \frac{2x}{x^2+1}$, so

$$f(-x) = \frac{2(-x)}{(-x)^2+1}$$

$$= -\frac{2x}{x^2+1}$$

$$= -f(x)$$

Hence, $f(x)$ is odd.

(e) (i) $\frac{d}{dx} xe^x = xe^x + e^x$

(ii) $\int \frac{d}{dx} xe^x \, dx = \int (xe^x + e^x) \, dx$

$$xe^x = \int xe^x \, dx + \int e^x \, dx$$

$$= \int xe^x \, dx + e^x$$

$\therefore \int xe^x \, dx = xe^x - e^x + C$ May leave constant out.

(f) (i) $Q = Q_0 e^{-kt} \rightarrow \frac{dQ}{dt} = -kQ_0 e^{-kt}$

$$= -kQ \quad //$$

(ii) At $t=0$, $Q = 100\text{mg}$, so $100\text{mg} = Q_0 \cdot e^0$

i.e. $Q_0 = 100\text{mg}$.

At $t = 20$ min, $Q = 74$ mg, so $74 = 100 e^{-20k}$

$$\log\left(\frac{74}{100}\right) = -20k$$

$$\therefore k = \frac{\log\left(\frac{74}{100}\right)}{-20}$$

$$\simeq 0.01506 \text{ min}^{-1}.$$

(iii) Since $\frac{dQ}{dt} = -kQ$,

when $t=0$, $Q = Q_0 = 100\,mg$.

Hence the initial rate of elimination is given by

$$\frac{dQ}{dt} = -kQ_0$$

$$= -(0.01506) \times 100\,mg \cdot min^{-1}$$

$$\simeq -1.5\ mg \cdot min^{-1}. \checkmark$$

(iv) We need t such that

$$50\,mg = 100\,mg \cdot e^{-kt}$$

so $$\frac{1}{2} = e^{-kt} \quad \checkmark$$

$$\log \tfrac{1}{2} = -kt$$

$$\therefore \quad t = \frac{\log 2}{k}$$

$$\simeq 46\ min. \quad \checkmark$$

QUESTION 16

(a)(i) In ΔPQS & ΔPRS:

(α) $\angle PQS = \angle PRS$ (given)
(β) $\angle PSQ = \angle PSR$ (given)
(γ) $PS = PS$ (common)

By angle-angle-side test, $\Delta PQS \equiv \Delta PRS$.

(ii) $QS = RS$ since they are corresponding sides of two congruent triangles.

(iii) $$\cos\theta = \frac{QS}{l}$$

$$QS = l\cos\theta$$

Since $QS = RS$, $QR = QS + SR$

$$= 2QS$$

$$= 2l\cos\theta \quad \text{, as required.}$$

(iv) Area, A, given by

$$A = \frac{1}{2} \cdot l \cdot QR \cdot \sin\theta$$

$$= \frac{1}{2} l \cdot 2l\cos\theta \sin\theta$$

$$= l^2 \cos\theta \sin\theta \quad //$$

(v) $\dfrac{dA}{d\theta} = \ell^2\left[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)\right]$

$= \ell^2\left[\cos^2\theta - \sin^2\theta\right]$

Extrema for θ : $\dfrac{dA}{d\theta} = 0$, so solve:

$$\ell^2(\cos^2\theta - \sin^2\theta) = 0$$

$$\sin^2\theta = \cos^2\theta$$

$$\tan^2\theta = 1$$

$$\therefore \tan\theta = \pm 1$$

implies $\theta = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$ for $0 \le \theta \le \pi$.

But $\theta \ne \dfrac{3\pi}{4}$ since angle sum of triangle would exceed π. So $\theta = \dfrac{\pi}{4}$ only.

Now, $\dfrac{d^2A}{d\theta^2} = \ell^2\left[2\cos\theta(-\sin\theta) - 2\sin\theta\cos\theta\right]$

$= -4\ell^2\cos\theta\sin\theta$

Evaluated at $\theta = \dfrac{\pi}{4}$ $\rightarrow$ $\dfrac{d^2A}{d\theta^2} = -4\ell^2 \cdot \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} < 0$

so $\theta = \dfrac{\pi}{4}$ is a minimum.

The maximum area is therefore,

$$A = \ell^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\ell^2}{2}.$$

(b) Let $A_0 = \$50\,000$, $M = \$1500$, $R = \left(1 + \dfrac{0.06}{12}\right) = 1.005$

(i) $A_1 = A_0R - M$

$= 50000(1.005) - 1500$

(ii) $A_2 = A_1R - M$

$= (A_0R - M)R - M$

$= A_0R^2 - MR - M$

$= 50\,000(1.005)^2 - 1500(1 + 1.005)$

$A_3 = A_2R - M$

$= \left[A_0R^2 - MR - M\right]R - M$

$= A_0R^3 - MR^2 - MR - M$

$= A_0R^3 - M(1 + R + R^2)$

$= 50\,000(1.005)^3 - 1500(1 + 1.005 + 1.005^2)$

(iii) The pattern developed in (ii) suggests for A_n:

$$A_n = A_0R^n - M(1+R+R^2+\cdots+R^{n-1})$$

$$= A_0R^n - M\cdot\frac{R^n-1}{R-1}, \text{ since } 1+R+R^2+\cdots+R^{n-1} \text{ is geometric.}$$

The account will reach zero for n such that $A_n=0$.

So solve for n in the following:

$$0 = A_0R^n - M\cdot\frac{R^n-1}{R-1}$$

$$= A_0(R-1)R^n - M(R^n-1)$$

$$= A_0(R-1)R^n - MR^n + M$$

$$= [A_0(R-1)-M]R^n + M$$

so

$$R^n = \frac{M}{M-A_0(R-1)}$$

$$n\log R = \log\left[\frac{M}{M-A_0(R-1)}\right]$$

$$n = \frac{\log\left[\frac{M}{M-A_0(R-1)}\right]}{\log R}$$

so $$n = \frac{\log\left[\frac{1500}{1500-50000\times0.005}\right]}{\log(1.005)}$$

$$= 36.5553\ldots$$

$$\doteqdot 37 \text{ months.}$$

(c) $3\sin^2\theta + 3\cos^2\theta + 3\tan^2\theta + 3\cot^2\theta + 3\sec^2\theta + 3\operatorname{cosec}^2\theta = 29$

$$3(\sin^2\theta + \cos^2\theta) + 3\tan^2\theta + 3\cot^2\theta + 3(1+\tan^2\theta) + 3(1+\cot^2\theta) = 29$$

$$\cancel{3} + 3\tan^2\theta + 3\cot^2\theta + \cancel{3} + 3\tan^2\theta + \cancel{3} + 3\cot^2\theta = \cancel{29}\ 20$$

$$6(\tan^2\theta + \cot^2\theta) = 20$$

$$3\tan^2\theta + \frac{3}{\tan^2\theta} = 10$$

$$3\tan^4\theta + 3 = 10\tan^2\theta$$

$$3\tan^4\theta - 10\tan^2\theta + 3 = 0$$

$$3\tan^4\theta - 9\tan^2\theta - \tan^2\theta + 3 = 0$$

$$3\tan^2\theta(\tan^2\theta - 3) - (\tan^2\theta - 3) = 0$$

$$(3\tan^2\theta - 1)(\tan^2\theta - 3) = 0$$

$$\therefore \tan^2\theta = \frac{1}{3} \text{ or } \tan^2\theta = 3$$

$$\text{so } \tan\theta = \pm\frac{1}{\sqrt{3}} \text{ or } \tan\theta = \pm\sqrt{3}$$

For $\tan\theta = \pm\frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$;

and

$\tan\theta = \pm\sqrt{3}$, $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

$\therefore \quad \theta \in \left\{ \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6} \right\}$.